Linear magnetohydrodynamic waves in a finely stratified plasma

Enzo Lazzaro,[1] Maurizio Lontano,[1] and Dmitri D. Ryutov[2]
[1]*Istituto di Fisica del Plasma ''Piero Caldirola,'' Consiglio Nazionale delle Ricerche, EURATOM-ENEA-CNR Association, Via R. Cozzi 53, 20125 Milano, Italy*
[2]*Lawrence Livermore National Laboratory, Livermore, California 94551*

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In a number of astrophysical systems, the magnetic field, instead of varying over a scale comparable with the ''natural'' scale of the object (e.g., tens of thousands of kilometers in the case of the solar convective zone), varies over lengths that are orders of magnitude less than this (e.g., over distances down to 100 km in the case of the magnetic filaments detected in the upper part of the solar convective zone and probably present in much deeper layers). Therefore, the study of the propagation of magnetohydrodynamic (MHD) waves in plasmas with fine magnetic nonuniformities is of considerable general importance for astrophysics. We have developed a general formalism that allows one to treat the propagation of large-scale MHD waves in a finely stratified medium. We demonstrate that the presence of a fine structure of the plasma may produce considerable modifications of the modes existing in a uniform plasma, with a number of propagation modes that may even increase. We also show that the slow MHD mode may experience a collisionless damping, which causes the wave energy to be converted into the energy of the peristaltic modes of the plasma ''resonant'' layers.

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I. INTRODUCTION

In a number of astrophysical objects, the magnetic field, instead of varying at a scale comparable with the ''natural'' scale of the object (e.g., tens of thousands of kilometers in the case of the solar convective zone), varies at scales that are orders of magnitude less than this ''natural'' scale (e.g., at a scale down to 100 km in the case of magnetic filaments detected in the upper part of the solar convective zone [1], and probably present in much deeper layers). Therefore, the study of the propagation of magnetohydrodynamic (MHD) waves in plasmas with fine magnetic nonuniformities is of considerable general importance for astrophysics.

The propagation of waves with a wavelength $\lambda\!\!\!^{-}$ considerably exceeding the scale length a of the nonuniformities is a difficult theoretical problem (here we mean the situation in general: not only MHD waves but also waves in objects like bubbly liquids, polycrystals, the turbulent atmosphere, etc.). Intuitively, it is tempting to say that the average (over a scale greater than the scale of nonuniformities) motion in the wave remains qualitatively the same as in the uniform medium, with the only difference that dispersion properties are now determined by some averaged characteristics of the medium. In part this is true (an acoustic wave in water with fine gas bubbles still remains an acoustic wave, although its dispersion may change considerably), but identification of the proper averaging procedure may be far from obvious. In addition, as we show in the present paper, there may also occur a considerable modification of the modes existing in the uniform system, and the number of modes may increase.

Moreover, the presence of the fine structures may also considerably increase the damping rate of long-wavelength perturbations. A good exposition of the physics of enhanced dissipation can be found in the text [2], where this problem is considered for acoustic waves in polycrystals.

In the past, the problem of MHD waves propagating in a plasma with fine nonuniformities has been considered in a number of papers [3–7]. In Refs. [3,4], the propagation of sound waves in a plasma with thin flux tubes was considered. The distance between adjacent flux tubes was assumed to be much greater than the typical flux-tube radius (''small filling factor''). The analysis was based on averaging of the dynamic equations. In Ref. [5] a multiple scattering technique was used to assess the sound propagation in a plasma with flux tubes all having the same parameters. These papers were based on the assumption of small filling factor.

In Ref. [6], the dispersion properties of waves were studied in the ''densely packed'' (large filling factor) case, where the size of the nonuniformities is comparable with the distance between them. To make the problem tractable, it was assumed that the relative variation of the plasma parameters is small compared to their average value. In [7], which is the closest predecessor of our paper, the dispersion of one-dimensional (1D) perturbations propagating across a finely stratified magnetic field (magnetosonic waves) was studied. We will give some further references to this paper in the appropriate parts of our study. References [6] and [7] also contain evaluations of the damping rate.

In this paper, we generalize the analysis of Ref. [7] to include waves propagating at an arbitrary angle to the magnetic field. As it turns out, this generalization is far from trivial: in contrast to the 1D case, where a magnetoacoustic wave in a finely structured medium maintains its identity and remains basically the same magnetoacustic wave but with the phase velocity determined by the averaged parameters of the medium, in our case identification of the modes with the modes of a uniform medium (Alfvén and magnetoacoustic) becomes impossible and, in addition, the number of eigenmodes may increase. Of course, the model of a stratified medium does not fully simulate the reality of structured fields, but it allows one to understand some features of such systems. The problem that we solve analytically in this paper may also become a test problem in developing computer codes addressing more general situations.

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In Sec. II, the geometry of the problem is described and the basic equations are derived. In Sec. III, an averaging procedure over the small scale a is introduced, and the ''average'' equations are presented. The distribution function of the plasma inhomogeneities is introduced in Sec. IV, and the possibility of getting a ''new'' root of the dispersion relation for MHD waves is demonstrated in Sec. V. The appearance of a collisionless damping of the slow MHD mode of the homogeneous case, induced by the small-scale structure, is shown in Sec. VI. Some general properties of the dispersion equation for long waves in finely structured plasmas are discussed in Sec. VII. Finally, Sec. VIII is devoted to concluding remarks.

II. THE GEOMETRY OF THE SYSTEM AND THE BASIC EQUATIONS

We assume that the unperturbed parameters of the system, the plasma density ρ_0, the plasma pressure p_0, and the magnetic field B_0, depend only on the coordinate x. The unperturbed magnetic field is directed along the axis z. The unperturbed total pressure

$$P_0 \equiv p_0 + \frac{B_0^2}{8\pi} \tag{1}$$

is uniform, $\partial P_0/\partial x = 0$.

The functions $\rho_0(x)$, $p_0(x)$, and $B_0(x)$ are some random functions of their argument (subject to the constraint that $P_0=\text{const}$). As two independent functions one can use the functions $\rho_0(x)$ and $B_0(x)$, or some functions of these variables. With regard to possible functional dependencies of $\rho_0(x)$ and $B_0(x)$, we assume that these functions vary by the order of unity with respect to their average values, and that their spatial dependence can be adequately characterized by a single parameter a having the dimension of a length; this parameter plays the role of the characteristic scale of the nonuniformities. The characteristic values of the gas-kinetic pressure $\rho_0(x)$ and the magnetic pressure $B_0^2/8\pi$ are assumed to be of the same order of magnitude. This means that the local values of the sound speed

$$V_s = \sqrt{\frac{\gamma p_0}{\rho_0}} \tag{2}$$

(where γ is the specific heat ratio) and the Alfvén speed

$$V_a = \sqrt{\frac{B_0^2}{4\pi\rho_0}} \tag{3}$$

are also of the same order of magnitude.

We consider small perturbations of this initial state. As it is a stationary state, one can seek for the perturbations in the form $\exp(-i\omega t)$. For such perturbations the standard set of linearized MHD equations reads as

$$-\omega^2\rho_0\boldsymbol{\xi} = -\nabla\,\delta p - \frac{1}{4\pi}\mathbf{B}_0\times\nabla\times\delta\mathbf{B} - \frac{1}{4\pi}\delta\mathbf{B}\times\nabla\times\mathbf{B}_0, \tag{4}$$

$$\delta\mathbf{B} = \nabla\times\boldsymbol{\xi}\times\mathbf{B}_0, \tag{5}$$

$$\delta\rho = -(\boldsymbol{\xi}\cdot\nabla)\rho_0 - \rho_0\nabla\cdot\boldsymbol{\xi}, \tag{6}$$

$$\frac{\delta p}{p_0} - \gamma\frac{\delta\rho}{\rho_0} + \boldsymbol{\xi}\cdot\left(\frac{\nabla\,\delta p}{p_0} - \gamma\frac{\nabla\,\delta\rho}{\rho_0}\right) = 0, \tag{7}$$

where $\boldsymbol{\xi}$ is a (small) displacement of the fluid element with respect to its unperturbed position, and the other perturbations are marked by the symbol δ. As is clear from this set of equations, we neglect all the dissipative processes.

As the system is uniform in the z direction, one can postulate an harmonic dependence of all the quantities on that coordinate. Accordingly, we assume that the z dependence of the perturbations has the form $\exp(iqz)$, with real q. Note that we assume that there is no y dependence of the perturbations.

For this type of perturbation, the y component of the displacement does not enter Eqs. (6) and (7). As for the y components of Eqs. (4) and (5), they reduce to

$$-\omega^2\rho_0\xi_y = \frac{iqB_0}{4\pi}\delta B_y, \tag{8}$$

$$\delta B_y = iqB_0\xi_y. \tag{9}$$

One sees that perturbations with this polarization are split from the perturbations with $\xi_y=0$. Equations (8) and (9) describe the shear Alfvén waves propagating along the z axis. The dispersion relation for this mode is

$$\omega^2 = \frac{q^2B_0^2}{4\pi\rho_0}. \tag{10}$$

As the Alfvén velocity, generally speaking, depends on x, Eq. (10) describes the so-called Alfvén continuum (see, e.g., Ref. [8]). We will not discuss this mode in our paper. We just recall that the presence of such solutions is a consequence of the fact that shear Alfvén waves have a zero group velocity across the magnetic field and, accordingly, these waves in separate layers propagate absolutely independently of each other.

Consider now equations where the x and z components of the displacement and of the perturbation of the magnetic field are present. After some algebra, these equations can be reduced to two equations for ξ_x and for the total pressure perturbation δP,

$$\delta P = \delta p + \frac{B_0\delta B_z}{4\pi}. \tag{11}$$

These equations read

$$\left(\omega^2\rho_0 - \frac{q^2B_0^2}{4\pi}\right)\xi_x = \frac{\partial\,\delta P}{\partial x}, \tag{12}$$

$$\frac{\partial\xi_x}{\partial x} = -\delta P\left(\frac{B_0^2}{4\pi} + \frac{\gamma p_0}{1-q^2\gamma p_0/\omega^2\rho_0}\right)^{-1}. \tag{13}$$

These are our basic equations that we will analyze in Sec. III. For the sake of reference, we present here the expressions of the other perturbations in terms of ξ_x:

$$\delta B_x = iqB_0\xi_x, \tag{14}$$

$$\delta B_z = -\frac{\partial}{\partial x}(\xi_x B_0), \tag{15}$$

$$\xi_z = \frac{iq\gamma p_0}{\gamma p_0 q^2 - \rho_0 \omega^2}\frac{\partial \xi_x}{\partial x}, \tag{16}$$

$$\delta p = -\xi_x \frac{\partial p_0}{\partial x} + \frac{\gamma p_0 \rho_0 \omega^2}{\gamma p_0 q^2 - \rho_0 \omega^2}\frac{\partial \xi_x}{\partial x}, \tag{17}$$

$$\delta \rho = -\xi_x \frac{\partial \rho_0}{\partial x} + \frac{\rho_0^2 \omega^2}{\gamma p_0 q^2 - \rho_0 \omega^2}\frac{\partial \xi_x}{\partial x}. \tag{18}$$

III. THE AVERAGED EQUATIONS

Equations (12)–(18) are, so far, exact equations. At this point we switch to the investigation of a particular class of the perturbations they describe, namely, perturbations with a scale length $ƛ$ much greater than the scale length a of micrononuniformities,

$$ƛ \gg a. \tag{19}$$

One can create such a perturbation by a very slow motion of the boundary of the plasma, with a frequency much less than the characteristic acoustic frequency of each slab,

$$\omega \ll V_s / a. \tag{20}$$

Alternatively, one can produce an initial perturbation by applying a smoothly varying ($ƛ \gg a$) external force, and then removing it and allowing the system to evolve freely. Obviously, the perturbation of the total pressure in every slab then remains almost uniform, as the perturbations under consideration represent slow quasistatic modes. Therefore, we conclude that the spatial dependence of the perturbation of the total pressure is a smooth function of x [see Fig. 1(a)].

In contrast, the perturbations of the gas-kinetic pressure and of the magnetic pressure, taken separately, can be considerable [Fig. 1(b)], because of the different compressibilities of the gas and the magnetic field. Also sharply varying will be ξ_z and the density perturbations. All these quantities adjust their variation in such a way as to keep the variation of the total pressure smooth.

The other perturbation that must be a smooth function of x is ξ_x. Indeed, if ξ_x varied by the order of unity at the scale a, then its spatial derivative entering Eq. (13) would be $\sim \xi_x / a$, the perturbation of the total pressure, according to Eq. (13), would be $\sim P_0 \xi_x / a$, and the right-hand side (rhs) of Eq. (12) would be formally much greater than the lhs, so that Eq. (12) could not be satisfied. Therefore, we conclude that ξ_x is also a smooth function of x.

To be more precise, one should say that, in the case of the functions ξ_x and δP, in addition to the smoothly varying component there are also jiggles, but their amplitude is many times less than the amplitude of the smoothly varying component [Fig. 1(a)]:

$$\xi_x = \langle \xi_x \rangle + \tilde{\xi}_x, \tag{21}$$

$$\delta P = \langle \delta P \rangle + \delta \tilde{P}, \tag{22}$$

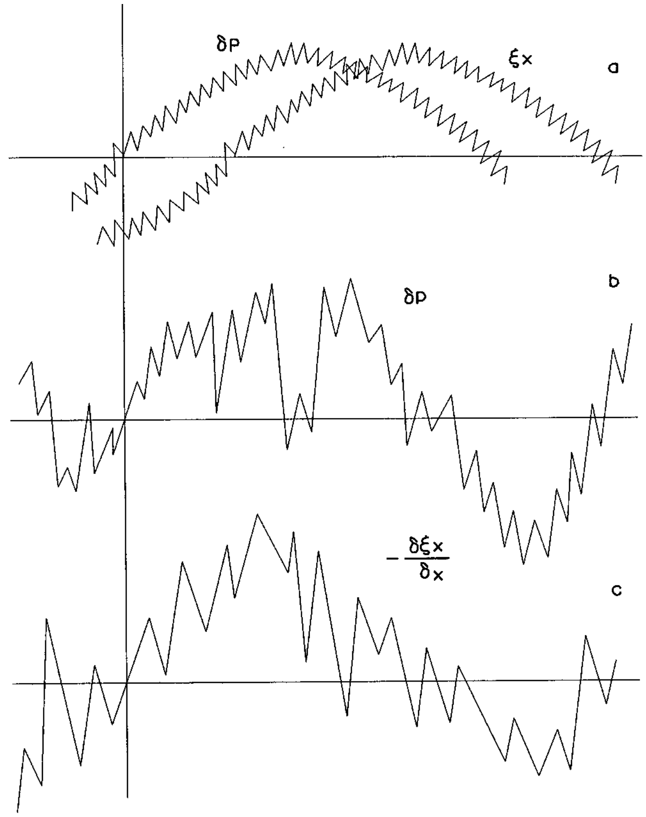

FIG. 1. Smoothly and sharply varying perturbations.

with

$$\tilde{\xi}_x \sim \langle \xi_x \rangle \frac{a}{ƛ}, \quad \delta \tilde{P} \sim \langle \delta P \rangle \frac{a}{ƛ}. \tag{23}$$

The averaging is carried out over a length l that is intermediate between a and $ƛ$,

$$a \ll l \ll \lambda, \tag{24}$$

and is defined as

$$\langle f(x) \rangle = \frac{1}{l}\int_{x-l/2}^{x+l/2} f(x')\,dx'. \tag{25}$$

In particular,

$$\begin{aligned}\left\langle \frac{\partial \delta P}{\partial x} \right\rangle &= \frac{1}{l}[\langle \delta P(x+l/2) \rangle - \langle \delta P(x-l/2) \rangle] \\ &\quad + \frac{1}{l}[\delta \tilde{P}(x+l/2) - \delta \tilde{P}(x-l/2)] \\ &= \frac{\partial \langle \partial P \rangle}{\partial x} + O\left(\frac{a}{l}\right) \approx \frac{\partial \langle \delta P \rangle}{\partial x}. \end{aligned} \tag{26}$$

The same result pertains to the averaging of $\partial \xi_x / \partial x$. Note also that the functions $\partial \xi_x / \partial x$ and $\partial \delta P / \partial x$ themselves (before averaging) are jittery functions [Fig. 1(c)]. This is immediately clear from Eqs. (12) and (13): these derivatives are products of smoothly varying functions and jittery functions and, therefore, are jittery.

Taking into account all these considerations, one can average Eqs. (12) and (13). Replacing [by virtue of Eqs. (21)–(23)] ξ_x in the lhs of Eq. (12) by $\langle \xi_x \rangle$, and δP in the rhs of Eq. (13) by $\langle \delta P \rangle$, one finds

$$\left(\omega^2 \langle \rho_0 \rangle - q^2 \left\langle \frac{B_0^2}{4\pi} \right\rangle \right) \langle \xi_x \rangle = \frac{\partial \langle \delta P \rangle}{\partial x}, \tag{27}$$

$$\frac{\partial \langle \xi_x \rangle}{\partial x} = -\langle \delta P \rangle \left\langle \frac{(\omega^2/q^2) - v_s^2}{\rho_0[(\omega^2/q^2)(v_a^2+v_s^2) - v_a^2 v_s^2]} \right\rangle. \tag{28}$$

Since the coefficients in these equations do not depend on x, one can look for a solution of the type exp(ikx) with real k. k plays the role of the x component of the wave number of the averaged perturbations. By introducing also the phase velocity of this perturbation,

$$v^2 = \frac{\omega^2}{k^2+q^2}. \tag{29}$$

and the angle ϑ formed by the total wave number with the axis z,

$$\cos^2 \vartheta = \frac{q^2}{k^2+q^2}, \tag{30}$$

one finds the following dispersion relation that allows one to express v in terms of ϑ:

$$\left(v^2 - \cos^2 \vartheta \frac{\langle \rho_0 v_a^2 \rangle}{\langle \rho_0 \rangle} \right) \left\langle \frac{\langle \rho_0 \rangle (v^2 - v_s^2 \cos^2 \vartheta)}{\rho_0[v^2(v_a^2+v_s^2) - \cos^2 \vartheta v_a^2 v_s^2]} \right\rangle = \sin^2 \vartheta. \tag{31}$$

In the case of a uniform plasma, one can delete the averaging signs and, in this way, arrive at the familiar dispersion relation describing fast and slow magnetosonic waves:

$$v^4 - v^2(v_a^2+v_s^2) + \cos^2 \vartheta v_a^2 v_s^2 = 0. \tag{32}$$

The other limiting case known from previous analyses [7] is that of $\vartheta = \pi/2$ (strictly perpendicular propagation). In this case, our dispersion relation is reduced to that of Ref. [7]:

$$v^2 = \frac{1}{\langle \rho_0 \rangle} \left\langle \frac{1}{\rho_0(v_a^2+v_s^2)} \right\rangle^{-1}. \tag{33}$$

In other words, in the case of cross-field propagation, the magnetoacoustic wave remains a magnetoacoustic wave, just with properly averaged phase velocity, but no new modes appear. In the more general case of the dispersion relation (31), as we will shortly show, the situation is much richer: the number of modes, generally speaking, increases, and in some cases a collisionless damping appears.

IV. THE DISTRIBUTION FUNCTION OF THE PLASMA NONUNIFORMITIES

Within the framework of the MHD approximation, the unperturbed state is characterized by three variables, p_0, ρ_0, and B_0. As there is the constraint $P_0 = \text{const}$, only two of them are independent. In order to characterize uniquely the initial state one can, therefore, use two of the aforementioned variables, or two functions of these variables. We will characterize the initial state by the values of the sound speed [Eq. (2)] and the Alfvén velocity [Eq. (3)]. Then, for instance, the density in the unperturbed state can be presented as

$$\rho_0 = \frac{P_0}{v_a^2/2 + v_s^2/\gamma}. \tag{34}$$

It is convenient to characterize the distribution of the nonuniformities of the unperturbed parameters v_a^2 and v_s^2 by the distribution function $F(v_a^2, v_s^2)$ (cf. [3]) defined as follows:

$$d\alpha = F(v_a^2, v_s^2) dv_a^2 dv_s^2, \tag{35}$$

where $d\alpha$ is the fraction of space occupied by the plasma whose parameters v_a^2 and v_s^2 lie in the range $dv_a^2 dv_s^2$. The normalization of the distribution function is obvious:

$$\int F(v_a^2, v_s^2) dv_a^2 dv_s^2 = 1. \tag{36}$$

Negative values of the parameters v_a^2 and v_s^2 are unphysical. Therefore distributions of the Gaussian type, which would extend to the negative values of these parameters, are obviously unphysical too. The function F should be nonzero only in one quadrant of the v_a^2, v_s^2 space. We will make an even stronger assumption that F is nonzero only in some range of the parameters v_a^2 and v_s^2 limited from both below and above.

The average [see Eq. (25)] of any function $f(v_a^2, v_s^2)$ can be conveniently expressed in terms of averaging over the distribution function F:

$$\langle f(v_a^2, v_s^2) \rangle = \int f(v_a^2, v_s^2) F(v_a^2, v_s^2) dv_a^2 dv_s^2. \tag{37}$$

For example, for the average density one has

$$\langle \rho_0 \rangle = P_0 \int \frac{F(v_a^2, v_s^2)}{v_a^2/2 + v_s^2/\gamma} dv_a^2 dv_s^2. \tag{38}$$

This allows one to express the dispersion relation in Eq. (31) in terms of averages over the distribution function F. We will not write this lengthy expression here.

Further simplifications occur in the case where the unperturbed temperature is uniform (this is an interesting case because the thermal diffusivity is usually much higher than the magnetic diffusivity and may quickly establish a uniform temperature over small scales). What is important to us is that, at a constant temperature, v_s^2 is constant, too, and the initial state is uniquely characterized by a single variable v_a^2. Accordingly, we need the distribution function only over one parameter, v_a^2. In this case the dispersion relation of Eq. (31) reads

$$\left(v^2-\cos^2\vartheta\frac{\langle\rho_0 v_a^2\rangle}{\langle\rho_0\rangle}\right)(v^2-v_s^2\cos^2\vartheta)\times\left\langle\frac{\langle\rho_0\rangle}{\rho_0[v^2(v_a^2+v_s^2)-\cos^2\vartheta v_a^2v_s^2]}\right\rangle=\sin^2\vartheta, \quad (39)$$

where

$$\langle\rho_0\rangle=P_0\int\frac{F(v_a^2)}{v_a^2/2+v_s^2/\gamma}dv_a^2, \quad (40)$$

$$\langle\rho_0 v_a^2\rangle=P_0\int\frac{v_a^2F(v_a^2)}{v_a^2/2+v_s^2/\gamma}dv_a^2, \quad (41)$$

$$\left\langle\frac{1}{\rho_0[v^2(v_a^2+v_s^2)-\cos^2\vartheta v_a^2v_s^2]}\right\rangle=\frac{1}{P_0}\int\frac{(v_a^2/2+v_s^2/\gamma)F(v_a^2)dv_a^2}{[v^2(v_a^2+v_s^2)-\cos^2\vartheta v_a^2v_s^2]}. \quad (42)$$

The possible presence of zeros in the denominator of the integral in Eq. (42) requires a proper treatment of this integral in the spirit of the usual treatment of Landau resonances [9]. That is, we first note that the integral in Eq. (42), if taken along the real axis of v_a^2, is an analytical function at $\mathrm{Im}\,\omega>0$; this is the function that enters the inverse Laplace transform (with $p=\mathrm{Im}\,\omega$) [9]. For negative values of $\mathrm{Im}\,\omega$, this function should be defined as an analytical continuation of the integral initially introduced. The role of the imaginary part of the rhs of Eq. (42) will be discussed in Secs. VI and VII.

V. NEW ROOTS OF THE DISPERSION EQUATION APPEAR

We will first analyze the dispersion Eq. (39) for a two-step distribution function, where the relevant parameter may acquire only two values, v_{a1}^2 and v_{a2}^2, randomly distributed over the space (see Fig. 2). The distribution function in this case is

$$F(v_a^2)=\alpha_1\delta(v_a^2-v_{a1}^2)+\alpha_2\delta(v_a^2-v_{a2}^2), \quad (43)$$

where $\alpha_1(\alpha_2)$ is the fraction of the space occupied by the first (the second) value of the parameter v_a^2, and

$$\alpha_1+\alpha_2=1. \quad (44)$$

In this case the dispersion relation takes the form

$$(v^2\langle\rho_0\rangle-\cos^2\vartheta\langle\rho_0v_a^2\rangle)(v^2-v_s^2\cos^2\vartheta)\times\left[\frac{\alpha_1(v_{a1}^2/2+v_s^2/\gamma)}{[v^2(v_{a1}^2+v_s^2)-\cos^2\vartheta v_{a1}^2v_s^2]}+\frac{\alpha_2(v_{a2}^2+v_s^2/\gamma)}{[v^2(v_{a2}^2+v_s^2)-\cos^2\vartheta v_{a2}^2v_s^2]}\right]=P_0\sin^2\vartheta. \quad (45)$$

It is convenient to rewrite Eq. (45) in terms of the following dimensionless variables:

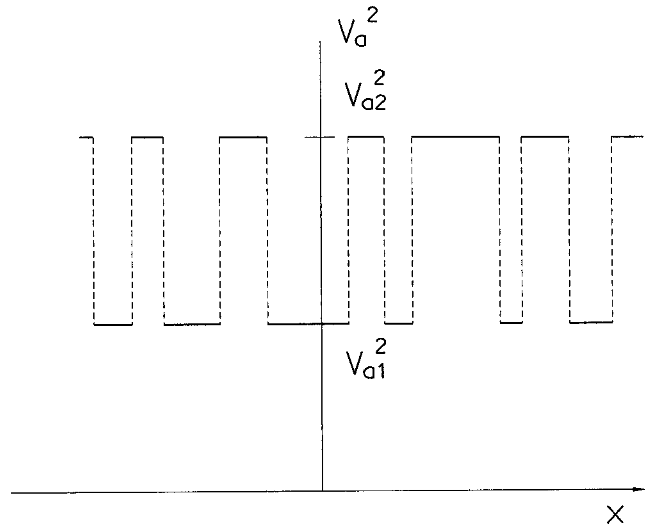

FIG. 2. Illustration of the two-value random function $v_a^2(x)$. The fraction of space ‘‘occupied’’ by the value v_{a1}^2 is α_1, the fraction of space occupied by v_{a2}^2 is α_2, and $\alpha_1+\alpha_2=1$.

$$u^2=\frac{v^2}{v_s^2\cos^2\vartheta},\quad u_{a1,2}^2=\frac{v_{a1,2}^2}{v_s^2}. \quad (46)$$

It then becomes

$$G(u)\equiv(Au^2-B)(u^2-1)\left[\frac{\alpha_1(u_{a1}^2/2+1/\gamma)}{[u^2(u_{a1}^2+1)-u_{a1}^2]}+\frac{\alpha_2(u_{a2}^2/2+1/\gamma)}{[u^2(u_{a2}^2+1)-u_{a2}^2]}\right]=\tan^2\vartheta, \quad (47)$$

where

$$A=\sum_{i=1,2}\frac{\alpha_i}{u_{ai}^2/2+1/\gamma},\quad B=\sum_{i=1,2}\frac{\alpha_iu_{ai}^2}{u_{ai}^2/2+1/\gamma}. \quad (48)$$

Before considering in more detail the case of two substances, we briefly discuss the reference case of a uniform medium, where $\alpha_1=1$ and $\alpha_2=0$. In this case the function G is reduced to a simpler function H that has the form

$$H(u^2)\equiv\frac{(u^2-u_a^2)(u^2-1)}{u^2(1+u_a^2)-u_a^2}. \quad (49)$$

A sketch of the function $H(u^2)$ is shown in Fig. 3 for $n=n_1=3\times10^8\ \mathrm{cm}^{-3}$, $T=100\ \mathrm{eV}$, $B=B_1=10\ \mathrm{G}$, $u_a^2=u_{a1}^2\approx100$, and $\beta=\beta_1\approx0.01$. The intersection of the plot of the function $H(u^2)$ with the horizontal line $H(u^2)=\tan^2\vartheta$ gives the values of the parallel phase velocity. The smaller of the roots

$$\frac{u_a^2}{1+u_a^2}<u^2<\min(1,u_a^2) \quad (50)$$

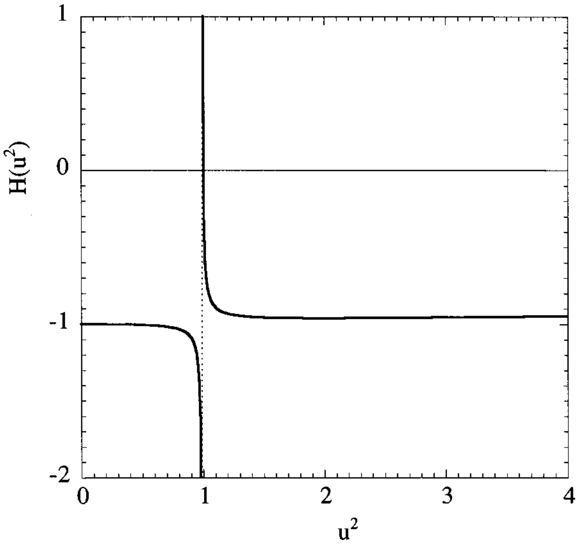

FIG. 3. The function $H(u^2)$ defined in Eq. (49) is plotted vs u^2, for $n=n_1=3\times10^8\ \mathrm{cm}^{-3}$, $T=100\ \mathrm{eV}$, $B=B_1=10\ \mathrm{G}$, $u_a^2=u_{a1}^2\approx100$, and $\beta=\beta_1\approx0.01$. Its intersection with the horizontal line $H(u^2)=\tan^2\vartheta$ gives the two solutions of the dispersion Eq. (47) in the case of a uniform plasma. Notice that the fast magnetoacoustic root is situated at high u^2 values which are outside the range displayed in the figure.

corresponds to the slow magnetoacoustic mode (see, for example, Ref. [10]). Its characteristic feature is that, in the limit $\vartheta\to\pi/2$, its parallel phase velocity becomes constant and tends to the value

$$u^2=\frac{u_a^2}{1+u_a^2}. \tag{51}$$

The larger of the roots

$$u^2>\max(1,u_a^2) \tag{52}$$

corresponds to the fast magnetoacoustic mode [10].

Consider now the case of two substances. Equation (47) is a third-order equation in u^2 [not a second-order equation as was the case in a uniform plasma; see Eq. (32)]. This points out the possible presence of an additional mode. One can show that, indeed, this equation universally has three solutions with positive (and thus physically meaningful) values of u^2 (unless we consider the degenerate case with $u_{a1}^2=u_{a2}^2$, which corresponds to the single-component system). Moreover, two asymptotes occur at

$$u^2=\frac{u_{a1(2)}^2}{1+u_{a1(2)}^2}. \tag{53}$$

The presence of the three roots is a direct consequence of the presence of the second component in the system.

For parallel propagation, i.e., for $\vartheta=0$, the three roots are $u^2=1$, $u^2=B/A$, and $u^2=u_{\mathrm{new}}^2$, where the ''new'' mode has the following dispersion:

TABLE I. Parameters for the two phases of a finely structured plasma.

	Phase $\alpha=1$	Phase $\alpha=2$
n_α (cm^{-3})	2.3×10^{10}	3×10^8
T_α (eV)	100	100
B_α (G)	3	10
$u_{a\alpha}^2$	≈0.1	≈100
β_α	≈10	≈0.01

$$u_{\mathrm{new}}^2=\frac{\dfrac{u_{a1}^2u_{a2}^2}{2}+\dfrac{\alpha_1u_{a2}^2}{\gamma}+\dfrac{\alpha_2u_{a1}^2}{\gamma}}{\dfrac{u_{a1}^2u_{a2}^2}{2}+\dfrac{1}{\gamma}+\alpha_1\left(\dfrac{u_{a2}^2}{\gamma}+\dfrac{u_{a1}^2}{2}\right)+\alpha_2\left(\dfrac{u_{a1}^2}{\gamma}+\dfrac{u_{a2}^2}{2}\right)}. \tag{54}$$

As a specific example, let us consider the two sets of parameters, shown in Table I, characterizing the two *phases* of the finely structured plasma under consideration. The total pressure P [see Eq. (1)] and the temperature T are assumed constant throughout the medium.

Three cases are now considered, depending on the relative concentration of the two phases of the plasma: $\alpha_1=\alpha_2=0.5$ [case (a)]; $\alpha_1=0.1$, $\alpha_2=0.9$ (b); $\alpha_1=0.9$, $\alpha_2=0.1$ (c). In all cases, $\gamma=\frac{5}{3}$ has been considered. In Fig. 4 the function $G(u^2)$ is plotted for the three cases (a), (b), and (c). Again, the intersection with the horizontal line $\tan^2\vartheta$ gives the three roots. The fast mode, modified by the presence of small-scale inhomogeneity, is the larger root of the dispersion, i.e., $\max(1,B/A)$. The smaller of $u^2=1$ and $u^2=B/A$ represents the modified slow magnetoacoustic mode. The root $u^2=u_{\mathrm{new}}^2$ is the new mode that can propagate inside the plasma due to its composite structure. Figure 5 shows the three roots of the dispersion Eq. (47) versus the propagation angle ϑ (in units of π), for the three cases (a), (b), and (c).

Both the two smaller roots have the characteristic signature of slow magnetoacoustic waves: a constant parallel phase velocity for $\vartheta\to\pi/2$. One can say that now there are two slow magnetoacoustic modes, instead of one as in the case of a uniform plasma. There exists also a fast mode, with phase velocity exceeding $\max(1,B/A)$.

Notice that in the two opposite limiting cases $\alpha_2\to0$ and $\alpha_1\to0$ the new root disappears, since the uniform plasma case should be recovered. In the former case u_{new} merges into the larger asymptote. In the latter case u_{new} merges into the lower asymptote. In both limits, u_{new} becomes independent of ϑ.

VI. COLLISIONLESS DAMPING OF THE SLOW MAGNETOACOUSTIC MODE

A second relevant example that allows a simple solution is that of an almost uniform medium, with nonuniformities occupying only a small fraction ϵ of the volume. In other words, we assume that the distribution function $F(v_a^2)$ has the form

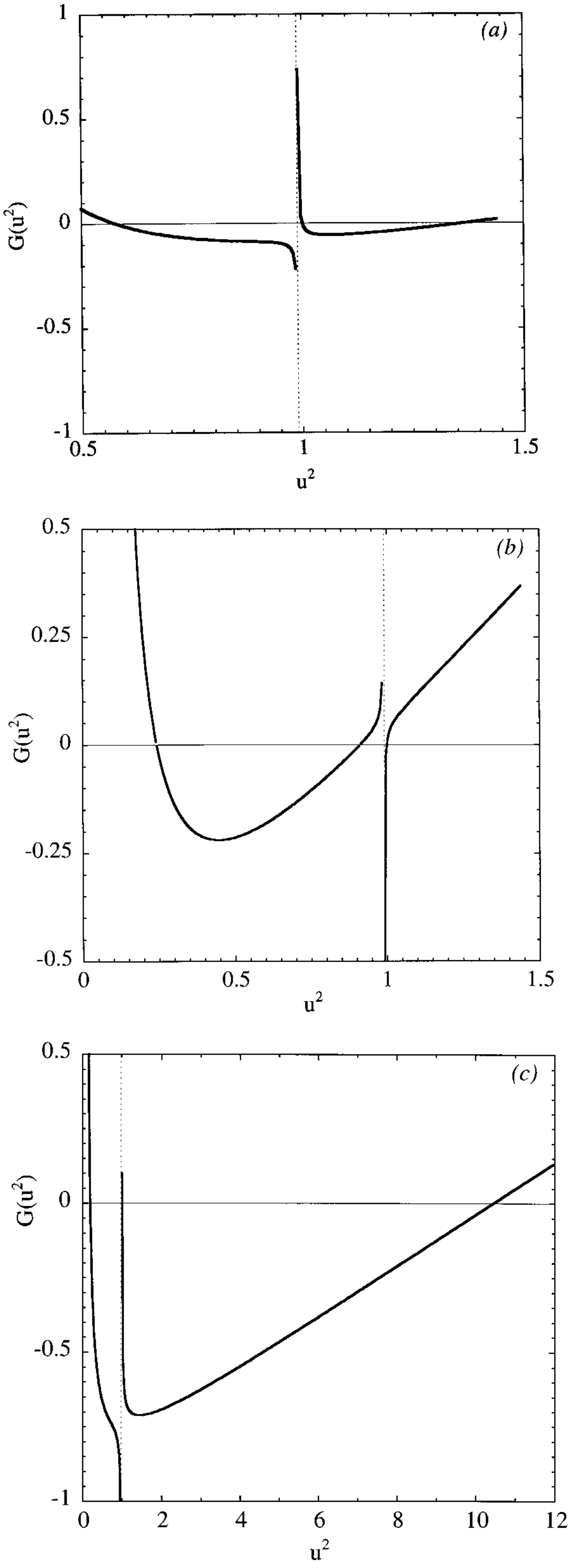

FIG. 4. The function $G(u^2)$ defined in Eq. (47) is plotted vs u^2, for three different values of the relative concentrations of the two phases of the plasma: $\alpha_1=\alpha_2=0.5$ (a); $\alpha_1=0.1$, $\alpha_2=0.9$ (b); $\alpha_1=0.9$, $\alpha_2=0.1$ (c). The other parameter values are those of Table I.

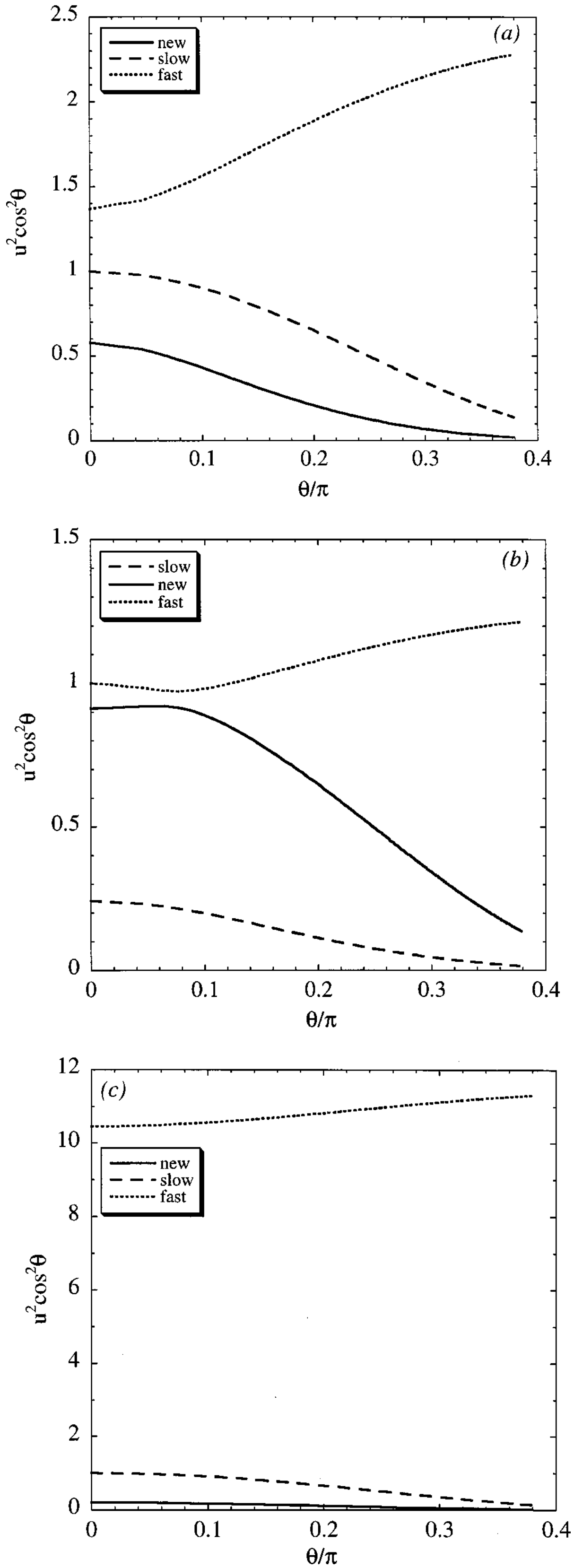

FIG. 5. The three roots of Eq. (47) are plotted as a function of the propagation angle ϑ (in units of π), for the same cases as in Fig. 4: $\alpha_1=\alpha_2=0.5$ (a); $\alpha_1=0.1$, $\alpha_2=0.9$ (b); $\alpha_1=0.9$, $\alpha_2=0.1$ (c). The ''new'' slow mode is plotted with a thicker line.

$$F(v_a^2)=(1-\epsilon)\,\delta(v_a^2-v_{a0}^2)+\epsilon f(v_a^2), \tag{55}$$

where the function $f(v_a^2)$ describes the presence of nonuniformities and is assumed to be normalized to the unity:

$\int f(v_a^2)dv_a^2=1$. When one substitutes the distribution function of Eq. (55) into the general Eq. (39), one finds the following dispersion relation pertaining to the case under consideration:

$$(Au^2-B)(u^2-1)\left[\frac{(1-\epsilon)(u_{a0}^2/2+1/\gamma)}{u^2(u_{a0}^2+1)-u_{a0}^2}+\epsilon\int\frac{f(u_a^2)(u_a^2/2+1/\gamma)du_a^2}{u^2(u_a^2+1)-u_a^2}\right]=\tan^2\vartheta. \quad (56)$$

Here, as in Eqs. (46) and (47), all the variables are normalized to the sound speed [Eq. (2)]. Moreover, we recall that we are considering the constant temperature case so the sound speed does not vary in the x direction.

The terms of the order of ε bring about two modifications to the solution of the uniform plasma case ($\varepsilon=0$): first, the real part of the frequency experiences a small shift proportional to ε; second, if there exists a range of phase velocities corresponding to the zeros of the denominator of the Landau integral in Eq. (56), there appears also an imaginary correction to the frequency. The first modification is of little interest and we do not consider it. The second modification introduces a qualitatively new element to the system—*collisionless damping*. Accordingly, in what follows, we retain only one term of the order ε in Eq. (56): the imaginary part of the Landau integral. In other words, we neglect the terms of order ε in A, B, and in the first term in the square bracket of Eq. (56), and arrive at the following approximate dispersion relation:

$$H(u^2)+i\varepsilon\nu\cong\tan^2\vartheta, \quad (57)$$

where the function H is defined according to Eq. (49), with u_{a0}^2 replacing u_a^2, and

$$\nu\equiv\frac{u^2-u_{a0}^2}{1/\gamma+u_{a0}^2/2}\,\mathrm{Im}\int_C\frac{1/\gamma+u_a^2/2}{u_a^2-u^2/(1-u^2)}f(u_a^2)du_a^2. \quad (58)$$

To be specific, consider a wave with k_z and $\mathrm{Re}\,\omega$ positive. In this case, for positive $\mathrm{Im}\,\omega$, one has $\mathrm{Im}\,u^2>0$. This determines the rule for treating the Landau integral in Eq. (58). The integration contour in the complex plane of u_a^2 should lie below the pole of the integrand. As, at small ε, the imaginary part of u^2 is small, one has (taking account of the integration rule just mentioned)

$$\mathrm{Im}\int_C\frac{1/\gamma+u_a^2/2}{u_a^2-u^2/(1-u^2)}f(u_a^2)du_a^2=\pi\int f(u_a^2)\left(\frac{1}{\gamma}+\frac{u_a^2}{2}\right)\delta\left(u_a^2-\frac{u^2}{1-u^2}\right)du_a^2. \quad (59)$$

The last integration is carried out along the real axis of u_a^2. The rhs of Eq. (59) is non-negative. It can be different from zero only for $u^2<1$ (otherwise, the argument of the δ function cannot become zero).

The presence of a small imaginary term in Eq. (57) gives rise to the appearance of a small imaginary part in u that can be found from the relationship

$$2u\,\mathrm{Im}\,u\,\frac{\partial H(u^2)}{\partial u^2}=-\varepsilon\nu, \quad (60)$$

where u should be determined from the solution of the unperturbed dispersion relation $H(u^2)=\tan^2\vartheta$.

The fast magnetoacoustic wave remains undamped, because its parallel phase velocity is always greater than 1 [see Eq. (52)]. Consider, therefore, the damping of slow magnetoacoustic waves. That this is a damping, not a growth, can easily be seen from inspection of the plot of $H(u^2)$ in Fig. 3, and the notion that, for slow modes, the parallel phase velocity is always below u_a [see Eq. (50)].

According to what has been said in Sec. IV, consider a distribution function that is different from zero in a finite interval, $u_{a,\min}<u_a<u_{a,\max}$. The damping of a slow wave with a certain phase velocity u occurs if u falls into the interval determined by the inequality

$$\frac{u_{a,\min}^2}{1+u_{a,\min}^2}<u^2<\frac{u_{a,\max}^2}{1+u_{a,\max}^2}<1. \quad (61)$$

As an example we have considered the following parabolic distribution function of the plasma nonuniformities:

$$f(u_a^2)=\frac{6(u_{a,\max}^2-u_a^2)(u_a^2-u_{a,\min}^2)}{(u_{a,\max}^2-u_{a,\min}^2)}H(u_{a,\max}^2-u_a^2)\times H(u_a^2-u_{a,\min}^2), \quad (62)$$

where the two Heaviside functions limit its definition to the finite interval $u_{a,\min}^2<u_a^2<u_{a,\max}^2$. The distribution is normalized to unity. In Fig. 6(a) the dispersion relation u^2 versus the angle ϑ (in units of π) is plotted for the case of the main plasma with parameters of the ''phase 2'' plasma (low β) of Sec. V. The range of existence of the distribution function is $u_{a,\min}^2=60$, $u_{a,\max}^2=200$. The fraction of the volume occupied by the nonuniformities is $\varepsilon=0.1$. According to Eq. (61) the imaginary part of u^2 exists in the range $0.983<u^2<0.995$, as shown in Fig. 6(b), where $\mathrm{Im}\,u^2$ is plotted vs u^2. Then the phase velocities which are affected by the collisionless damping lie below the dashed line in Fig. 6(a). It is seen that the collisionless damping of the slow mode becomes effective for propagation sufficiently oblique to the direction of the external magnetic field (the z axis). It is consistent with the nature of damping that lies in the resonant interaction between the external waves and slow perturbations propagating in the plasma layers (along the magnetic field) with a proper u_a. This damping mechanism is the same as that discussed in Ref. [4] (see also the survey [11]).

VII. GENERAL PROPERTIES OF THE DISPERSION RELATION (39)

In this section we consider the general dispersion Eq. (39). When written in dimensionless quantities u^2 and u_a^2 defined as in Eq. (46), this dispersion relation reads

$$(Au^2-B)(u^2-1)\int_C\frac{(u_a^2/2+1/\gamma)F(u_a^2)du_a^2}{u^2(u_a^2+1)-u_a^2}=\tan^2\vartheta. \quad (63)$$

The distribution function is normalized to 1. As before, we assume that F differs from zero only within a finite interval of u_a^2.

Equation (63) possesses an interesting property: it predicts the presence of undamped fast magnetoacoustic modes

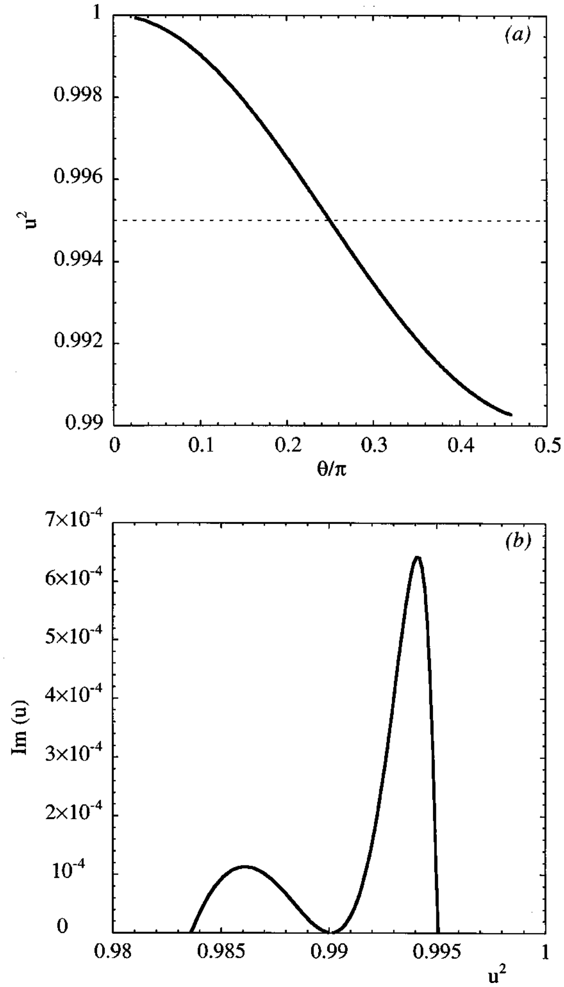

FIG. 6. The root of Eq. (49) corresponding to the slow mode in a uniform plasma is plotted vs ϑ (a). The relevant imaginary part, of collisionless origin, is plotted vs u^2 (b), for $u_{a,\min}^2=60$, $u_{a,\max}^2=200$, as discussed in Sec. VI.

with phase velocities satisfying the condition $u^2>\max(1,B/A)$. The parallel phase velocity of the fast magnetoacoustic mode is greater than the zero of the denominator of the integrand in Eq. (63)—whence the absence of damping. A simple asymptotic solution of Eq. (63) exists for $\vartheta\approx\pi/2$. In this case the parallel phase velocity of the fast mode is very large, and Eq. (63) reduces to

$$u^2\int\frac{F(u_a^2)du_a^2}{u_a^2/2+1/\gamma}\int\frac{(u_a^2/2+1/\gamma)F(u_a^2)du_a^2}{u_a^2+1}=\tan^2\vartheta. \tag{64}$$

The slow magnetoacoustic mode, generally speaking, experiences a significant damping, with $\mathrm{Im}\,u\approx\mathrm{Re}\,u$.

VIII. SUMMARY AND DISCUSSION

We have developed a general formalism that allows one to treat the propagation of large-scale MHD waves in a finely stratified medium. We have obtained equations that allow one to gain some insight into the wave properties of a plasma with a large filling factor, where nonuniformities are tightly packed to each other.

On the one hand, our equations predict the occurrence of new roots in addition to those of the uniform plasma case. For the particular plasma nonuniformity distribution considered here [corresponding to a two-phase plasma; see Eq. (43)], a new slow wave exists in addition to the standard slow and fast magnetoacoustic modes. The more complicated, although more realistic, situation of a continuously varying Alfvén velocity inside the plasma, generally speaking, involves zeros of trascendental functions which may give rise to multiple solutions; it will be the matter of future investigation.

On the other hand, it has been demonstrated that in this latter case, for a small degree of nonuniformity, the standard slow mode, generally speaking, experiences collisionless damping: the wave energy is converted into energy of the peristaltic modes of the resonant layers, i.e., the layers where the phase velocity of the peristaltic modes coincides with the parallel phase velocity of the wave. Generally speaking, the damping of the slow mode is strong, with the imaginary and real parts of the frequency being of the same order of magnitude. A transparency window for slow modes may be present in the system if the range of Alfvén velocities where the distribution function is different from zero does not entirely overlap with the range of phase velocities of the slow mode.

The fast mode is universally undamped, as its parallel phase velocity is always above the phase velocity of the peristaltic modes.

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